

First Derivative Test for Local Extrema

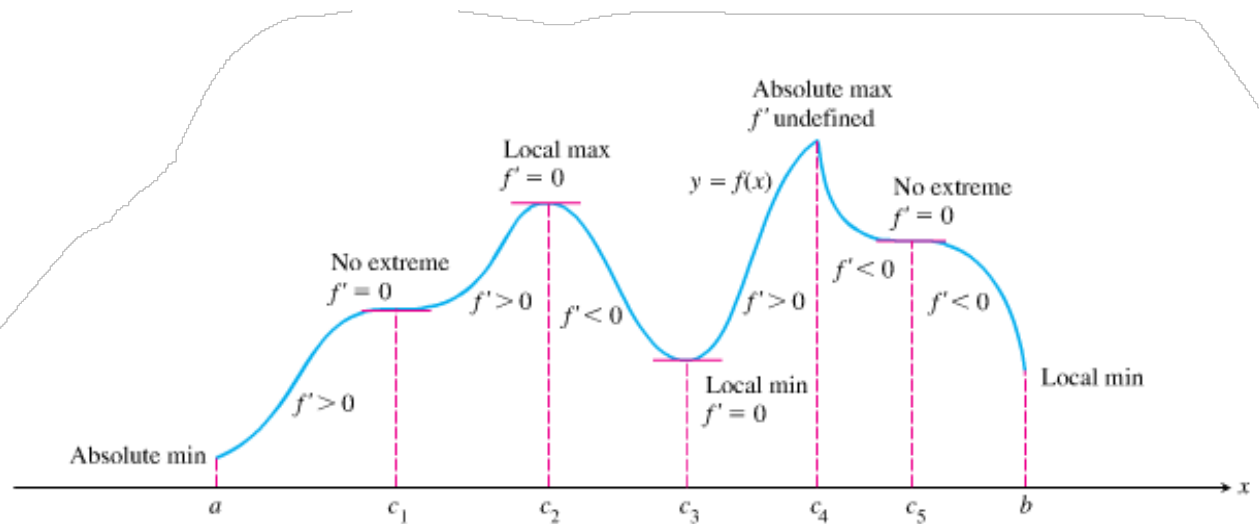


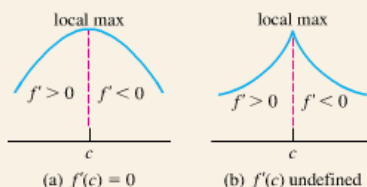
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

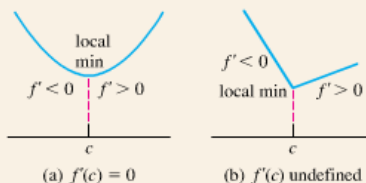
At a critical point c :

1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c .

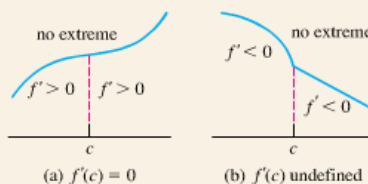


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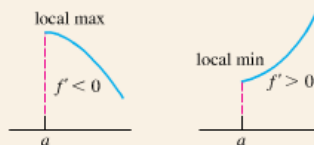
2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c .



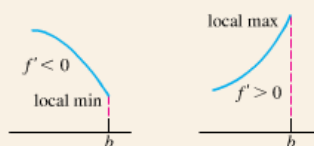
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c .

**At a left endpoint a :**

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a .

**At a right endpoint b :**

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b .



EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$

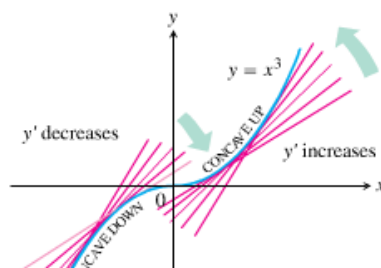
(b) $g(x) = (x^2 - 3)e^x$

Concavity

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

- (a) **concave up** on an open interval I if y' is increasing on I .
- (b) **concave down** on an open interval I if y' is decreasing on I .



Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

- (a) concave up on any interval where $y'' > 0$.
- (b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$

(b) $y = 3 + \sin x$ on $(0, 2\pi)$

$$\begin{aligned} \text{a) } y' &= 2x \\ y'' &= 2 \end{aligned}$$

always concave
up because $y'' > 0$

$(3, 10)$ concave

$$\begin{aligned} \text{b) } y' &= \cos x \\ y'' &= -\sin x \end{aligned} \quad +$$

Concave down $(0, \pi)$
because $y'' < 0$.

concave up $(\pi, 2\pi)$
because $y'' > 0$.

Points of Inflection

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

$$y' = e^{-x^2} (-2x) \\ = -2x e^{-x^2}$$

$$y'' = -2x(-2x e^{-x^2}) + (-2)(e^{-x^2})$$

$$= 2e^{-x^2} (2x^2 - 1)$$

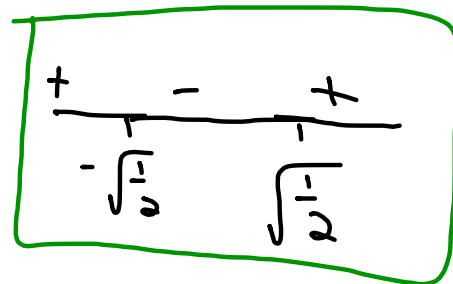
never
= 0
or undefined

$$2x^2 - 1 = 0$$

$$x^2 = \frac{1}{2} \\ x = \pm \sqrt{\frac{1}{2}}$$

$$\left(-\sqrt{\frac{1}{2}}, e^{-1/2}\right)$$

$$\left(\sqrt{\frac{1}{2}}, e^{-1/2}\right)$$



points of inflection because the sign of y'' changes

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f , justifying each answer with information obtained from the graph of f' .

- (a) On what intervals is f increasing? $[-4, -2), (-2, 1)$ since $f' > 0$
- (b) On what intervals is the graph of f concave up? $(-2, 0), (3, 4)$
because f' is increasing
- (c) At which x -coordinates does f have local extrema?
 $x = -4, x = 1, x = 4$ (endpoints and where $f' = 0$ and there is a sign change)
- (d) What are the x -coordinates of all inflection points of the graph of f ?
 $x = -2, x = 0, x = 3$ because f' has a max or min
- (e) Sketch a possible graph of f on the interval $[-4, 4]$.

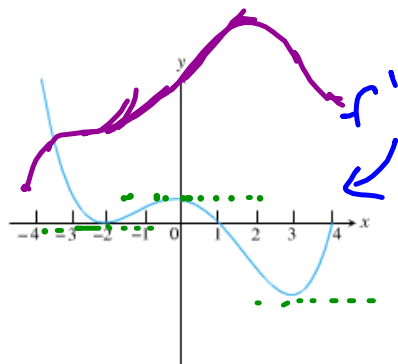


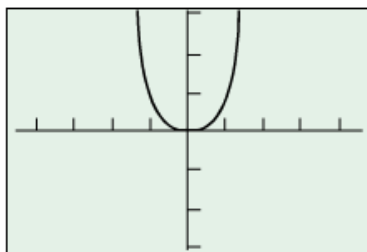
Figure 4.25 The graph of f' , the derivative of f , on the interval $[-4, 4]$.

possible f

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

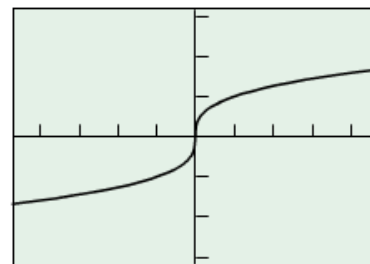
1. *The second derivative can be zero at a noninflection point.* For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point.* For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical tangent line*, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x -axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

$$v(t) = 6t^2 - 28t + 22$$

$$= 2(3t^2 - 14t + 11) \quad \begin{array}{c} + \quad - \quad + \\ \hline 0 \quad 1 \quad 11/3 \end{array}$$

$$= 2(3t - 11)(t - 1)$$

$$= 0 \text{ when } t = 1 \quad t = 11/3$$

Changes direction at $t = 1$
and $t = 11/3$

$$a(t) = 12t - 28$$

$$= 4(3t - 7)$$

$$t = \frac{7}{3}$$

$$\begin{array}{c} - \quad + \\ \hline 1 \quad 7/3 \end{array}$$

Speed is increasing $(11/3, \infty) (1, 7/3)$
because the signs
of $v(t)$ and $a(t)$
are the same

Second Derivative Test for Local Extrema

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

$$f'(x) = 3x^2 - 12 \quad \begin{array}{l} -8 + 24 - 5 \\ 8 - 24 - 5 \end{array}$$

$$= 3(x^2 - 4)$$

$$= 3(x+2)(x-2)$$

$$= 0 \text{ at } x = -2 \text{ } x = 2$$

$$f''(x) = 6x$$

$$f''(-2) = -12 \quad \therefore f''(x) < 0$$

max at $x = -2$

$$f''(2) = 12 \quad f''(x) > 0$$

min at $x = 2$

local max of 11

local min of -21

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

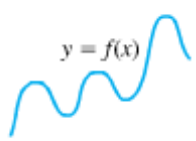

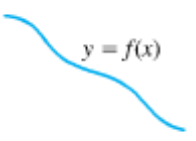




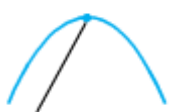

- (a) Identify where the extrema of f occur.
- (b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
- (c) Find where the graph of f is concave up and where it is concave down.
- (d) Sketch a possible graph for f .

EXPLORATION 1 Finding f from f'

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives

 <p>$y = f(x)$</p> <p>Differentiable \Rightarrow smooth, connected; graph may rise and fall</p>	 <p>$y = f(x)$</p> <p>$y' > 0 \Rightarrow$ graph rises from left to right; may be wavy</p>	 <p>$y = f(x)$</p> <p>$y' < 0 \Rightarrow$ graph falls from left to right; may be wavy</p>
 <p>or</p> <p>$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall</p>	 <p>or</p> <p>$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall</p>	 <p>y'' changes sign</p> <p>Inflection point</p>
 <p>or</p> <p>y' changes sign \Rightarrow graph has local maximum or minimum</p>	 <p>$y' = 0$ and $y'' < 0$ at a point; graph has local maximum</p>	 <p>$y' = 0$ and $y'' > 0$ at a point; graph has local minimum</p>

EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f' , with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x -coordinates of all local extrema and points of inflection of f .

(b) Sketch a possible graph of f .

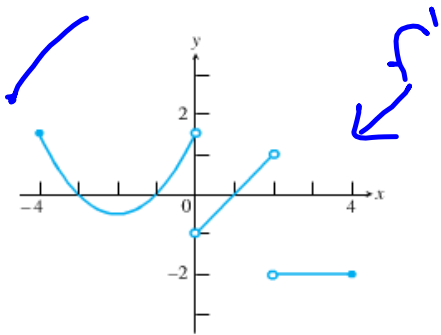


Figure 4.33 The graph of f' , a discontinuous derivative.

a) extremes :
 $x = -3$ max $x = -4$ min
 $x = -1$ min $x = 4$ min
 $x = 0$ max
 $x = 1$ min
 $x = 2$ max
 points of inflection :
 $x = -2$

EXPLORATION 2 Finding f from f' and f''

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	+	does not exist	-	0	-
f''	+	does not exist	+	0	-

- Find where all absolute extrema of f occur. *absolute max at $x = 0$*
- Find where the points of inflection of f occur. *at $x = 2$*
- Sketch a possible graph of f .

